Homogenizing the Darcy/Stokes coupling

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Abstract We study a fluid flow traversing a porous medium and obeying the Darcy's law in the case when this medium is fractured by a periodical distribution of fissures filled with a Stokes fluid. These two flows are coupled by a Beavers-Joseph type interface condition. As the small period of the distribution shrinks to zero, the resulting asymptotic behaviour is implicitely described by two underlying macroscopic quantities: the limit of the Stokes velocity and the limit of the Darcy pressure, solutions of a new coupled system obtained by homogenization. The behaviour of the macroscopic two-scale limit of the filtration velocity is given by an explicit two-scale Darcy type law, presenting coupling terms with the gradient of the limit of the Darcy pressure and with the limit of the Stokes velocity.

Keywords Fractured porous media · Stokes flow · Beavers-Joseph interface · Homogenization · Two-scale convergence

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1 Introduction

We consider an incompressible viscous fluid flow in a periodically structured domain consisting of two interwoven regions, separated by an interface. The first region represents the system of fissures which form the connected fracture, where the viscous flow is governed by the Stokes system. The second region, which may be also connected, stands for a porous structure of a certain permeability, where the flow is governed by Darcy's law. These two flows are coupled on the interface by the Saffman's variant [15] of the Beavers-Joseph condition [6], [10] which was confirmed by [9] as the limit of a homogenization process. Besides the continuity of the normal component of the velocity, it imposes the proportionality of

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the tangential velocity with the tangential component of the viscous stress on the fissure-side of the interface.

Modelling of fractured porous media(see [4], [5], [17], [8] and [14]) is addressed as a macroscopic phenomenon emerging from the alteration of an homogeneous porous medium by a distribution of microscopic fissures. As the process happens at a microscopic scale, under the assumption of ε-periodicity, the study of its asymptotic behaviour is amenable to the procedures of the homogenization theory. The porous part of the material is assumed to obey the homogenized Darcy's law which is already a macroscopic approximation of a microscopic process. Therefore, the question arises as to which extent the porous part may be considered as existing besides the Stokes flow. The answer will be brougth in the form of an original model where it is assumed that, at least on a microscopic level, both media, the porous one and the fluid one, share the same orders of magnitude.

One major achievement of homogenization theory was the mathematical justification of Darcy's law [9] based on arguments from the homogenization of perforated domains with isolated holes [7], that is, the homogenized structure stems from a domain which, on a microscopic scale, is not connected. It was not until this assumption could be dropped out that the homogenization of phenomena in fractured media could be accomplished(see [12], [3] and [13]).

The paper is organized as follows.

Section 2 is devoted to the mathematical modelisation of the fractured material, namely, the slip boundary condition of Beavers-Joseph type is revisited in accordance with observed physical laws and the underlying arguments of the classical mathematical theory. In that respect, the usual nondimensional constants are introduced in (2.5) and rescaled thanks to (2.23)–(2.24) so that it becomes relevant to consider the asymptotic behaviour of the family of problems indexed by the little parameter ε.

The homogenization process is studied in Section 3 in the case where the scaling parameter β in (2.23)–(2.24) is zero, which is actually the most involving one. The mathematical study of Section 3 addapts the methods of the two-scale convergence theory(see [2] and [11]). The a priori estimate (3.2) serves as the departure point for the existence of the limiting procedure which is described in Section 3 as a three-term balance between the limits u, v and p arising in (3.24), (3.25) and (3.45) respectively, therefore leading to a two-component asymptotic velocity (u, v) instead of the original u^ε. This singular behaviour reflects the coexistence of one macroscopic level featured out by the macroscopic pair (v, p) which lives in the N-dimensional open set Ω on one hand, and the microscopic term u which lives in the product $\Omega \times Y$, that is, the original open set Ω augmented by an N-dimensional set containing the microscopic dual variable.

2 Preliminaries

Let Ω be an open connected bounded set in $\mathbb{R}^N (N \geq 2)$, locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_f be a Lipschitz open connected subset of $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$, such that if we repeat $\overline{Y}$ by periodicity, the reunion of all $\bar{Y}_f$-parts is a connected set in $\mathbb{R}^N$

with a boundary which have the C^2-regularity property [1]. Denoting it by $\mathbb{R}^N_f$ we introduce

$$Y_s = Y \setminus \mathbb{R}^N_f, \quad \Gamma = \partial Y_f \cap \partial Y_s. \tag{2.1}$$

We assume that the measures of Y_f and Y_s are strictly positive.

For any $\varepsilon \in]0,1[$, we define the regions which represent the system of fissures and the porous structure by:

$$\Omega_{\varepsilon f} = \Omega \cap (\varepsilon \mathbb{R}^N_f), \quad \Omega_{\varepsilon s} = \Omega \setminus \overline{\Omega_{\varepsilon f}}. \tag{2.2}$$

Their interface is denoted by

$$\Gamma_\varepsilon = \partial \Omega_{\varepsilon f} \cap \partial \Omega_{\varepsilon s}. \tag{2.3}$$

Let us remark that $\Omega_{\varepsilon f}$ is connected, that $\Omega_{\varepsilon s}$ can be connected as well and that the porosity of our fractured structure is represented by

$$m := |Y_f| \in]0,1[, \quad \text{as} \quad \frac{|\Omega_{\varepsilon f}|}{|\Omega|} \to m \quad \text{when} \quad \varepsilon \to 0. \tag{2.4}$$

If K^ε, μ^ε and α^ε_{BJ} stand for the positive tensor of permeability in $\Omega_{\varepsilon s}$, the viscosity of the fluid in $\Omega_{\varepsilon f}$ and the dimensionless Beavers-Joseph coefficient on Γ_ε, then, by denoting

$$A^\varepsilon = \frac{\varepsilon^{2\beta}}{\mu_\varepsilon}(K^\varepsilon)^{-1} \quad (\beta \geq 0), \quad \alpha_\varepsilon = \frac{\varepsilon^{\beta-1}\alpha^\varepsilon_{BJ}}{\mu_\varepsilon} \in C^1(\overline{\Omega}) \tag{2.5}$$

the Stokes-Darcy system, coupled by the Saffman's variant of the Beavers-Joseph condition, takes the form:

$$\text{div} u^\varepsilon = 0 \quad \text{in} \quad \Omega \tag{2.6}$$

$$A^\varepsilon u^\varepsilon + \nabla p^\varepsilon = g^\varepsilon \quad \text{in} \quad \Omega_{\varepsilon s}, \quad A^\varepsilon \in (L^\infty(\Omega))^{N^2} \tag{2.7}$$

$$-\varepsilon^{2\beta}\Delta u^\varepsilon + \nabla p^\varepsilon = g^\varepsilon \quad \text{in} \quad \Omega_{\varepsilon f}, \quad g^\varepsilon \in (L^2(\Omega))^N \tag{2.8}$$

$$[u^\varepsilon_n]_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad n \text{ is the outward normal on } \partial\Omega_{\varepsilon f}, \tag{2.9}$$

$$[p^\varepsilon]_\varepsilon n + \varepsilon^{2\beta}\frac{\partial u^\varepsilon}{\partial n} = \varepsilon\alpha_\varepsilon(\gamma_{\varepsilon f} u^\varepsilon - u^\varepsilon_n n) \quad \text{on} \quad \Gamma_\varepsilon, \tag{2.10}$$

$$u^\varepsilon_n = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_{\varepsilon s} \tag{2.11}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_{\varepsilon f} \tag{2.12}$$

where u^ε and p^ε stand for the velocity and the pressure in our fractured porous media, where $\gamma_{\varepsilon f}$ is the trace operator corresponding to $H^1(\Omega_{\varepsilon f})$ and where, for any $\varphi \in H(\text{div}, \Omega)$, (or $H(\text{div}, \Omega)^N$), we use the notation

$$[\varphi_n]_\varepsilon = \varphi_n|_{\Omega_{\varepsilon s}} - \varphi_n|_{\Omega_{\varepsilon f}} \quad \text{on} \quad \Gamma_\varepsilon. \tag{2.13}$$

Denoting

$$H_0(\text{div}, \Omega) = \{\zeta \in L^2(\Omega)^N, \quad \text{div}\zeta = 0 \quad \text{in} \quad \Omega, \quad \zeta_n = 0 \quad \text{on} \quad \partial\Omega\} \tag{2.14}$$

$$V_\varepsilon = \{\zeta \in H^1(\Omega_{\varepsilon f})^N, \quad \zeta = 0 \quad \text{on} \quad \partial\Omega \cap \Omega_{\varepsilon f}\} \tag{2.15}$$

we introduce the Hilbert space

$$H_\varepsilon = \{\zeta \in H_0(\mathrm{div}, \Omega), \quad \zeta|_{\Omega_{\varepsilon f}} \in V_\varepsilon\} \tag{2.16}$$

endowed with the scalar product

$$\langle u, \zeta\rangle_{H_\varepsilon} = \int_{\Omega_{\varepsilon s}} u\cdot\zeta + \varepsilon^{2\beta}\int_{\Omega_{\varepsilon f}} \nabla u \nabla \zeta + \varepsilon \int_{\Gamma_\varepsilon} (u - u_n n)(\zeta - \zeta_n n), \quad \forall u, \zeta \in H_\varepsilon. \tag{2.17}$$

The variational formulation of (2.6)–(2.12) follows.

To find $u^\varepsilon \in H_\varepsilon$ such that

$$\int_{\Omega_{\varepsilon s}} A^\varepsilon u^\varepsilon \zeta + \varepsilon^{2\beta}\int_{\Omega_{\varepsilon f}} \nabla u^\varepsilon \nabla \zeta + \varepsilon \int_{\Gamma_\varepsilon} \alpha_\varepsilon (u^\varepsilon - u_n^\varepsilon n)(\zeta - \zeta_n n) = \int_\Omega g^\varepsilon \zeta, \quad \forall \zeta \in H_\varepsilon. \tag{2.18}$$

Using the result proved in the scalar case by [3], we obtain immediately:

Lemma 1 *There exists $C > 0$, independent of ε, such that*

$$|\zeta|_{L^2(\Omega_{\varepsilon f})} \le C |\nabla \zeta|_{L^2(\Omega_{\varepsilon f})}, \quad \forall \zeta \in V_\varepsilon. \tag{2.19}$$

Thus, we see that the Lax-Milgram theorem can be applied and hence:

Theorem 1 *There exists a unique $u^\varepsilon \in H_\varepsilon$ solution of (2.18).*

The asymptotic behaviour of u^ε and p^ε, when $\varepsilon \to 0$, will be studied under the following hypotheses.

$$\exists A \in L^\infty_{\mathrm{per}}(Y)^{N^2} \text{ positively defined such that } A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \ x \in \Omega \tag{2.20}$$

$$\exists \alpha \in C^1_{\mathrm{per}}(Y) \text{ and } \quad \alpha_0 > 0 \text{ such that } \alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right) \ge \alpha_0, \ x \in \Omega \tag{2.21}$$

$$\exists g \in L^2(\Omega)^N \quad \text{such that} \quad g^\varepsilon \to g \quad \text{strongly in} \quad L^2(\Omega). \tag{2.22}$$

Assuming that K^ε has all the eigenvalues of the same order with respect to ε, and denoting it by $O(K^\varepsilon)$, we see that in fact we are in the case when

$$O(\mu^\varepsilon) O(K^\varepsilon) = O(\varepsilon^{2\beta}) \tag{2.23}$$

$$O(\alpha^\varepsilon_{BJ}) O(K^\varepsilon) = O(\varepsilon^{\beta+1}) \tag{2.24}$$

3 The homogenization process when $\beta = 0$

Setting $\zeta = u^\varepsilon$ in (2.18) we get

$$|u^\varepsilon|^2_{H_\varepsilon} \leq C|u^\varepsilon|_{L^2(\Omega)} \tag{3.1}$$

Applying (2.19) we find that

$$\{u^\varepsilon\}_\varepsilon \text{ is bounded in } H_\varepsilon \text{ and in } H_0(\mathrm{div}, \Omega). \tag{3.2}$$

$$|u^\varepsilon|_{H^1(\Omega_{\varepsilon f})} \leq C, \quad C \text{ being independent of } \varepsilon \tag{3.3}$$

It follows that $\exists \hat{u} \in L^2(\Omega \times Y)^N$ such that, on some subsequence

$$u^\varepsilon \stackrel{2s}{\rightharpoonup} \hat{u} \quad \text{(two-scale) in} \quad L^2(\Omega, C_{\mathrm{per}}(Y))^N \tag{3.4}$$

$$u^\varepsilon \rightharpoonup \int_Y \hat{u}(\cdot, y)dy \in H_0(\mathrm{div}, \Omega) \quad \text{weakly in} \quad L^2(\Omega)^N \tag{3.5}$$

Denoting $\chi_{\varepsilon f}(x) = \chi_f\left(\frac{x}{\varepsilon}\right)$ and $\chi_{\varepsilon s}(x) = \chi_s\left(\frac{x}{\varepsilon}\right)$, where χ_f and χ_s are the characteristic functions of Y_f and Y_s in Y, we see that $(\chi_{\varepsilon s}u^\varepsilon)_\varepsilon$, $(\chi_{\varepsilon f}u^\varepsilon)_\varepsilon$ ad $\left(\chi_{\varepsilon f}\frac{\partial u^\varepsilon}{\partial x_i}\right)_\varepsilon$ are bounded in $(L^2(\Omega))^N$, $\forall i \in \{1, 2, \cdots, N\}$.

It follows that $\exists \eta_i \in L^2(\Omega \times Y)^N$ such that, on some subsequence of (3.4)–(3.5) we have also

$$\chi_{\varepsilon s}u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_s\hat{u} \quad \text{(two-scale) in} \quad L^2(\Omega, C_{\mathrm{per}}(Y))^N \tag{3.6}$$

$$\chi_{\varepsilon f}u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_f\hat{u} \quad \text{(two-scale) in} \quad L^2(\Omega, C_{\mathrm{per}}(Y))^N \tag{3.7}$$

$$\chi_{\varepsilon f}\frac{\partial u^\varepsilon}{\partial x_i} \stackrel{2s}{\rightharpoonup} \eta_i \quad \text{(two-scale) in} \quad L^2(\Omega, C_{\mathrm{per}}(Y))^N \tag{3.8}$$

Denoting by

$$H^1_{\mathrm{per}}(Y_f) = \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^N_f), \quad \varphi \quad \text{is } Y\text{-periodic}\} \tag{3.9}$$

we can present a first result.

Lemma 2 *There exist $v \in H^1_0(\Omega)^N$ and $w \in L^2(\Omega, (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N)$ such that*

$$\hat{u}|_{\Omega \times Y_f} = v \tag{3.10}$$

$$\eta_i = \chi_f\left(\frac{\partial v}{\partial x_i} + \frac{\partial w}{\partial y_i}\right) \tag{3.11}$$

Proof. Let $\psi \in \mathcal{D}(\Omega, C^{\infty}_{\rm per}(Y_f))$ with $\displaystyle\int_Y \psi = 0$ in Ω. Let us consider $\varphi \in \mathcal{D}(\Omega, H^1_{\rm per}(Y_f)^N)$ satisfying

$$\mathrm{div}_y \varphi = \psi \quad \text{in} \quad \Omega \times Y_f \tag{3.12}$$

$$\varphi_n = 0 \quad \text{in} \quad \Omega \times \Gamma. \tag{3.13}$$

Defining $\varphi^\varepsilon(x) = \varphi\left(x, \dfrac{x}{\varepsilon}\right)$ we find that $\varphi^\varepsilon \in H^1(\Omega_{\varepsilon f})^N$ and $\varphi^\varepsilon_n = 0$ on Γ_ε. As $u^\varepsilon \in H^1(\Omega_{\varepsilon f})^N$ with $u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_{\varepsilon f}$ it follows

$$\int_{\Omega_{\varepsilon f}} u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx =$$

$$= -\varepsilon \int_{\Omega_{\varepsilon f}} \frac{\partial u^\varepsilon}{\partial x_i}(x)\varphi_i\left(x, \frac{x}{\varepsilon}\right) dx - \varepsilon \int_{\Omega_{\varepsilon f}} u^\varepsilon(x)(\mathrm{div}_x \varphi)\left(x, \frac{x}{\varepsilon}\right) dx. \tag{3.14}$$

Passing at the limit on the subsequence on which (3.4)–(3.8) hold, we find

$$\int_{\Omega \times Y_f} u(x,y)\psi(x,y)dxdy = 0. \tag{3.15}$$

It follows that $\exists v \in L^2(\Omega)^N$ such that $u|_{\Omega \times Y_f} = v$ and hence

$$\chi_{\varepsilon f} u^\varepsilon \rightharpoonup mv \quad \text{weakly in} \quad L^2(\Omega)^N. \tag{3.16}$$

But from Lemma A.3 [3] we know that $\exists \hat{v} \in H^1_0(\Omega)$ such that, by extracting a subsequence of (3.16) we have

$$\chi_{\varepsilon f} u^\varepsilon \rightharpoonup m\hat{v} \quad \text{weakly in} \quad L^2(\Omega)^N, \tag{3.17}$$

that is $v = \hat{v} \in H^1_0(\Omega)$.
It remains to prove (3.11). First, we remark that $\eta_i|_{\Omega \times Y_s} = 0$. Now, let $\psi \in \mathcal{D}(\Omega; H^1_{\rm per}(Y_f)^N)$ such that

$$\mathrm{div}_y \psi = 0 \quad \text{in} \quad \Omega \times Y_f \tag{3.18}$$

$$\psi_n = 0 \quad \text{on} \quad \Omega \times \Gamma. \tag{3.19}$$

It follows that $\psi^\varepsilon(x) = \psi\left(x, \dfrac{x}{\varepsilon}\right)$ has the properties:

$$\psi^\varepsilon \in H^1(\Omega_{\varepsilon f})^N \quad \text{and} \quad \psi^\varepsilon_n = 0 \quad \text{on} \quad \Gamma_\varepsilon \tag{3.20}$$

$$\int_{\Omega_{\varepsilon f}} \frac{\partial u^\varepsilon}{\partial x_i}(x)\psi^\varepsilon_i(x)dx = -\int_{\Omega_{\varepsilon f}} u^\varepsilon(x)(\mathrm{div}_x \psi)\left(x, \frac{x}{\varepsilon}\right) dx. \tag{3.21}$$

Using the two-scale convergences (3.4)–(3.8) we find

$$\int_{\Omega \times Y_f} \eta_i(x,y)\psi_i(x,y)dxdy = -\int_\Omega v(x)\mathrm{div}_x\left(\int_{Y_f} \psi(x,y)dy\right) dx =$$

$$= \int_{\Omega \times Y_f} \frac{\partial v}{\partial x_i}(x)\psi_i(x,y)dxdy \tag{3.22}$$

and the proof is completed. ■

From now on we use the notation

$$u = \hat{u}|_{\Omega \times Y_s} \in L^2(\Omega \times Y_s). \tag{3.23}$$

In the light of Lemma 1 and of the new notations we sum the convergence results obtained until now.

Theorem 2 *There exist $u \in L^2(\Omega\times Y_s)$, $v \in H_0^1(\Omega)$ and $w \in L^2(\Omega, (H^1_{\rm per}(Y_f)/\mathbb{R})^N)$ such that the following convergences hold on some subsequence.*

$$\chi_{\varepsilon s} u^\varepsilon \overset{2s}{\rightharpoonup} \chi_s u \quad \text{two-scale in} \quad L^2(\Omega, C_{\rm per}(Y))^N \tag{3.24}$$

$$\chi_{\varepsilon f} u^\varepsilon \overset{2s}{\rightharpoonup} \chi_f v \quad \text{two-scale in} \quad L^2(\Omega, C_{\rm per}(Y))^N \tag{3.25}$$

$$\chi_{\varepsilon f}\frac{\partial u^\varepsilon}{\partial x_i} \overset{2s}{\rightharpoonup} \chi_f\left(\frac{\partial v}{\partial x_i}+\frac{\partial w}{\partial y_i}\right) \text{ two-scale in } L^2(\Omega, C_{\rm per}(Y))^N, \ \forall i\in\{1,\cdots,N\} \tag{3.26}$$

Moreover, by denoting

$$\tilde u(x) = \frac{1}{|Y_s|}\int_{Y_s} u(x,y)dy \quad \text{for a.e. } x\in\Omega \tag{3.27}$$

we have

$$\tilde u_n = 0 \quad \text{on} \quad \partial\Omega \tag{3.28}$$

$$(1-m)\,{\rm div}\tilde u + m\,{\rm div} v = 0 \quad \text{in} \quad \Omega \tag{3.29}$$

$${\rm div}_y u = 0 \quad \text{in} \quad \Omega\times Y_s \tag{3.30}$$

$${\rm div}_y w + {\rm div} v = 0 \quad \text{in} \quad \Omega\times Y_f \tag{3.31}$$

$$u(x,y)\cdot n(y) = v(x)\cdot n(y) \quad \text{for a.e.} \quad (x,y)\in\Omega\times\Gamma. \tag{3.32}$$

Proof. The convergences (3.24)–(3.26) are straight consequences of (3.6)–(3.8) and Lemma 2. The properties (3.28)–(3.29) follow from (3.5).
Now let $\varphi \in \mathcal{D}(\Omega, H^1_{\rm per}(Y_s))$.
Defining

$$\varphi^\varepsilon(x) = \begin{cases} \varepsilon\varphi\left(x, \dfrac{x}{\varepsilon}\right) & \text{for a.e. } x\in\Omega_{\varepsilon s} \\ 0 & \text{for a.e. } x\in\Omega_{\varepsilon f}\end{cases} \tag{3.33}$$

We see that $\varphi^\varepsilon \in \mathcal{D}(\Omega)$ and hence

$$\begin{aligned} 0 = &\int_\Omega u^\varepsilon \nabla\varphi^\varepsilon(x)dx = \\ =&\int_\Omega \chi_{\varepsilon s}(x)u^\varepsilon(x)(\nabla_y\varphi)\Big(x,\frac{x}{\varepsilon}\Big)dx + \varepsilon\int_\Omega \chi_{\varepsilon s}(x)u^\varepsilon(x)(\nabla_x\varphi)\Big(x,\frac{x}{\varepsilon}\Big)dx. \end{aligned} \tag{3.34}$$

Using (3.24), we pass to the limit and get

$$\int_{\Omega\times Y_s} u(x,y)\nabla_y\varphi(x,y)dxdy = 0 \tag{3.35}$$

and (3.30) is proved. The relation (3.31) can be obtained similarly.
Finally, let $\varphi\in\mathcal{D}(\Omega, H^1_{\rm per}(Y))$. denoting

$$\varphi^\varepsilon(x) = \varepsilon\varphi\left(x,\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x\in\Omega, \tag{3.36}$$

we see that $\varphi^\varepsilon\in H^1_0(\Omega)$. Using ${\rm div}(u^\varepsilon)=0$ in Ω we obtain easily

$$\int_\Omega (\chi_{\varepsilon s}u^\varepsilon)(x)(\nabla_y\varphi)\left(x,\frac{x}{\varepsilon}\right)dx + \int_\Omega (\chi_{\varepsilon f}u^\varepsilon)(x)(\nabla_y\varphi)\left(x,\frac{x}{\varepsilon}\right)dx = O(\varepsilon) \tag{3.37}$$

Using (3.24)–(3.25) we find that

$$\int_{\Omega\times Y_s} u(x,y)(\nabla_y\varphi)(x,y)dxdy + \int_{\Omega\times Y_f} v(x)(\nabla_y\varphi)(x,y)dxdy = 0 \tag{3.38}$$

which implies

$$\int_{\Omega\times\Gamma}(v(x)n(y)-u(x,y)n(y))\varphi(x,y)dxdy=0 \tag{3.39}$$

and the proof is completed. ■

Now, in order to study the asymptotic behaviour of u^ε, we have to recover the pressure hidden by the variational formulation, and to obtain corresponding convergence properties.

First, we set $\zeta\in H_\varepsilon$ in (2.18) such that

$$\zeta\in H_0^1(\Omega)^N \quad \text{and} \quad \zeta=0 \quad \text{in} \quad \Omega_{\varepsilon f}. \tag{3.40}$$

Using the classical L^2 decomposition, we find that $\exists p^{\varepsilon s}\in H^1(\Omega_{\varepsilon s})$ such that

$$\nabla p^{\varepsilon s}=A^\varepsilon u^\varepsilon-g^\varepsilon\in L^2(\Omega_{\varepsilon s}) \tag{3.41}$$

$$\int_{\Omega_{\varepsilon s}}(-p^{\varepsilon s}\mathrm{div}\varphi+A^\varepsilon u^\varepsilon\varphi)+\int_{\Gamma_\varepsilon}p^{\varepsilon s}n\varphi=\int_{\Omega_{\varepsilon s}}g^\varepsilon\varphi, \quad \forall\varphi\in(H_0^1(\Omega))^N. \tag{3.42}$$

Moreover, using the celebrated inequality (see [18]):

$$|\theta|_{L^2(\Omega_{\varepsilon s})}\leq C\varepsilon|\nabla\theta|_{L^2(\Omega_{\varepsilon s})}, \quad \forall\theta\in H_0^1(\Omega_{\varepsilon s})^N \tag{3.43}$$

we find that $\exists C>0$, independent of ε, such that:

$$|\nabla p^{\varepsilon s}|_{H^{-1}(\Omega_{\varepsilon s})}\leq C\varepsilon. \tag{3.44}$$

Thus, the hypotheses of Theorem 3.2 [13] are fullfilled and we have

Theorem 3 *There exists $p\in H^1(\Omega)/\mathbb{R}$ such that, on some sub-subsequence of that of Theorem 2, the following convergence holds*

$$p^{\varepsilon s} \text{ (extended with 0)} \quad \stackrel{2s}{\rightharpoonup} \quad \chi_s p \quad \text{(two-scale) in} \quad L^2(\Omega\times Y). \tag{3.45}$$

Next, we set $\zeta\in H_\varepsilon$ in (2.18) such that

$$\zeta\in H_0^1(\Omega)^N \quad \text{and} \quad \zeta=0 \quad \text{in} \quad \Omega_{\varepsilon s}. \tag{3.46}$$

It follows that $\exists p^{\varepsilon f}\in L^2(\Omega_{\varepsilon f})$ such that

$$\langle\nabla p^{\varepsilon f},\psi\rangle_{H^{-1}(\Omega_{\varepsilon f})}=\int_{\Omega_{\varepsilon f}}(\nabla u^\varepsilon\nabla\psi-g^\varepsilon\psi), \quad \forall\psi\in H_0^1(\Omega_{\varepsilon f})^N \tag{3.47}$$

For any $i\in\{1,2,\cdots,N\}$, let us define $T_i^{\varepsilon f}\in L^2(\Omega_{\varepsilon f})^N$ by

$$(T_i^{\varepsilon f})_j=-p^{\varepsilon f}\delta_{ij}+\frac{\partial u_i^\varepsilon}{\partial x_j} \tag{3.48}$$

From (3.47) we get successively

$$-\mathrm{div}T_i^{\varepsilon f}=g_i^\varepsilon\in L^2(\Omega_{\varepsilon f}) \tag{3.49}$$

$$\int_{\Omega_{\varepsilon f}}(-p^{\varepsilon f}\mathrm{div}\varphi+\nabla u^\varepsilon\nabla\varphi)-\int_{\Gamma_\varepsilon}(T_i^{\varepsilon f}n)\varphi_i=\int_{\Omega_{\varepsilon f}}g^\varepsilon\varphi, \quad \forall\varphi\in H_0^1(\Omega)^N. \tag{3.50}$$

Adding (3.50) to (3.42) and comparing with (2.18) we obtain

$$p^{\varepsilon s} n_i - T_i^{\varepsilon f} n = \varepsilon \alpha_\varepsilon (u_i^\varepsilon - u_n^\varepsilon n_i) \quad \text{in} \quad H^{-1/2}(\Gamma_\varepsilon), \quad \forall i, \tag{3.51}$$

which is the weak formulation of the Beavers-Joseph condition. Thus, we have proved that u^ε, $p^{\varepsilon s}$ and $p^{\varepsilon f}$ verify

$$\int_{\Omega_{\varepsilon s}} (-p^{\varepsilon s} \mathrm{div}\varphi + A^\varepsilon u^\varepsilon \varphi) + \int_{\Omega_{\varepsilon f}} (-p^{\varepsilon f} \mathrm{div}\varphi + \nabla u^\varepsilon \nabla \varphi) + \tag{3.52}$$

$$+\varepsilon \int_{\Gamma_\varepsilon} \alpha_\varepsilon (u^\varepsilon - u_n^\varepsilon n)(\varphi - \varphi_n n) = \int_\Omega g^\varepsilon \varphi, \quad \forall \varphi \in H_0^1(\Omega)^N.$$

By density (3.52) holds for any $\varphi \in W_\varepsilon$ where

$$W_\varepsilon = \{\psi \in H(\mathrm{div}, \Omega), \quad \psi_n = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_{\varepsilon s}, \quad \psi|_{\Omega_{\varepsilon f}} \in V_\varepsilon\}. \tag{3.53}$$

It is obvious that, leaving a real constant aside, we can suppose that $p^\varepsilon \in L^2(\Omega)$, defined by

$$p^\varepsilon = \begin{cases} p^{\varepsilon s} \text{ in } \Omega_{\varepsilon s} \\ p^{\varepsilon f} \text{ in } \Omega_{\varepsilon f} \end{cases} \tag{3.54}$$

has zero mean on Ω. It follows that there exists $\varphi^\varepsilon \in H_0^1(\Omega)^N$ such that

$$\mathrm{div}\varphi^\varepsilon = p^\varepsilon \quad \text{in} \quad \Omega \tag{3.55}$$

$$|\nabla \varphi^\varepsilon|_{L^2(\Omega)} \le C |p^\varepsilon|_{L^2(\Omega)} \quad \text{with } C \text{ independent of } \varepsilon \tag{3.56}$$

For any $\varphi \in H_0^1(\Omega)^N$, we have like in [8]:

$$\varepsilon^{1/2} |\varphi|_{L^2(\Gamma_\varepsilon)} \le C \left(\varepsilon |\nabla \varphi|_{L^2(\Omega_{\varepsilon f})} + |\varphi|_{L^2(\Omega_{\varepsilon f})} \right). \tag{3.57}$$

Combining it with (2.19) we get, via (3.57):

$$\varepsilon \int_{\Gamma_\varepsilon} (\varphi^\varepsilon - \varphi_n^\varepsilon n)^2 \le C |\nabla \varphi^\varepsilon|^2_{L^2(\Omega_{\varepsilon f})} \le C |p^\varepsilon|^2_{L^2(\Omega)}. \tag{3.58}$$

Then, puting $\varphi = \varphi^\varepsilon$ in (3.52) we obtain immediately

$$|p^\varepsilon|_{L^2(\Omega)} \le C, \quad \text{for some } C > 0 \text{ independent of } \varepsilon. \tag{3.59}$$

It yields the following result:

Lemma 3 *There exists $q \in L^2(\Omega \times Y)$ with zero mean on Y such that*

$$p^{\varepsilon f} \textit{(extended with zero)} \overset{2s}{\rightharpoonup} \chi_f q \quad \textit{(two-scale) in} \quad L^2(\Omega \times Y). \tag{3.60}$$

Finally, we shall pass (3.52) to the limit using a very special family of test functions. The limit relation will describe the behaviour of u, v, w and p as the unique solution of a certain system.

Theorem 4 *For any* $\theta \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y_s)^N)$, $\psi \in \mathcal{D}(\Omega, (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N)$ *and* $\varphi \in \mathcal{D}(\Omega)^N$ *having the properties*

$$\mathrm{div}_y \psi = -\mathrm{div}\varphi \quad in \quad \Omega \times Y_f, \tag{3.61}$$

$$\mathrm{div}_y \theta = 0 \quad in \quad \Omega \times Y_s, \tag{3.62}$$

the following relation holds:

$$\int_{\Omega\times Y_s} (-p\,\mathrm{div}_x\theta + Au\theta) + \int_{\Omega\times Y_f} (\nabla v + \nabla_y w)(\nabla\varphi + \nabla_y\psi) + \tag{3.63}$$
$$+ \int_{\Omega\times\Gamma} \alpha(v - v_n n)\varphi = \int_{\Omega\times Y} g(\chi_s\theta + \chi_f\varphi).$$

Proof. Denoting $\psi^\varepsilon(x) = \psi\left(x, \dfrac{x}{\varepsilon}\right)$, $\theta^\varepsilon(x) = \theta\left(x, \dfrac{x}{\varepsilon}\right)$, $x \in \Omega$, we define

$$\zeta_\varepsilon = (\chi_{\varepsilon s}\theta^\varepsilon + \chi_{\varepsilon f}\varphi + \varepsilon\chi_{\varepsilon f}\psi^\varepsilon) \in W_\varepsilon. \tag{3.64}$$

Seting $\zeta = \zeta^\varepsilon$ in (3.52) and using the two-scale convergences (2.22), (3.24)–(3.26), (3.45) and (3.60) we obtain (3.63).
We present here only the convergence of the term involving Γ_ε, the other being straightforward.
Let $\varphi \in \mathcal{D}(\Omega)^N$; for any $i, j \in \{1, 2, \cdots, N\}$, there exists $G_{ij} \in \mathcal{D}(\Omega, C^1_{\mathrm{per}}(Y_f))$ such that

$$G_{ij}(x, y) = (\varphi_i(x) - \varphi_k(x)n_k(y)n_i(y))n_j(y) \quad \text{on} \quad \Gamma. \tag{3.65}$$

This follows from the C^1-property on Γ of the right-hand side of (3.65) and from the smoothness of its prolongation with zero on ∂Y_f. Thus, $G^\varepsilon_{ij}(\cdot) = G_{ij}\left(\cdot, \frac{\cdot}{\varepsilon}\right) \in C^1_0(\Omega_{\varepsilon f})$ and we have

$$\varepsilon\int_{\Gamma_\varepsilon} \alpha_\varepsilon(u^\varepsilon - u^\varepsilon_n n)(\varphi - \varphi_n n) = \varepsilon\int_{\partial\Omega_{\varepsilon f}} \alpha_\varepsilon G^\varepsilon_{ij} u^\varepsilon_i n^\varepsilon_j d\sigma =$$
$$= \int_{\Omega_{\varepsilon f}} \frac{\partial(\alpha G_{ij})}{\partial y_j}\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon_i(x)dx + O(\varepsilon)$$
$$\to \int_\Omega v(x)\left(\int_{Y_f} \frac{\partial(\alpha G_{ij})}{\partial y_j}(x, y)dy\right)dx = \int_{\Omega\times\Gamma} \alpha v(\varphi - \varphi_n n) \tag{3.66}$$

and the proof is completed. ∎

We introduce the main local-solutions of our study. Denoting

$$V_f = \{\varphi \in (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N,\ \mathrm{div}_y\varphi = 0\}, \tag{3.67}$$

$$K_f = \{\varphi \in (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N,\ \mathrm{div}_y\varphi = -1\} \tag{3.68}$$

we define $V^{kh} \in V_f$, $k, h \in \{1, 2, \cdots, N\}$ and $W \in K_f$ as the unique solutions of the problems:

$$\int_{Y_f} \left(\delta_{ik}\delta_{jh} + \frac{\partial V^{kh}_i}{\partial y_j}\right)\frac{\partial\psi_i}{\partial y_j} = 0, \quad \forall\psi \in V_f. \tag{3.69}$$

$$\int_{Y_f} \nabla W \nabla\psi = 0, \quad \forall\psi \in V_f. \tag{3.70}$$

For (3.69) the Lax-Milgram Theorem yields the existence and uniqueness results by using the following Poincaré inequality:

$$|\psi|_{L^2(Y_f)} \leq C|\nabla\psi|_{L^2(Y_f)}, \quad \forall\psi \in (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N. \tag{3.71}$$

As regarding (3.70) we see that W is in fact the projection of 0 on the closed convex $K_f \neq \emptyset$ in $(H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N$.

By setting $\theta = \varphi = 0$ in (3.63) we see that as $v \in H^1_0(\Omega)^N$ is given, then the problem

$$\int_{\Omega\times Y_f} (\nabla v + \nabla_y w)\nabla_y \psi = 0, \quad \forall\psi \in L^2(\Omega, V_f) \tag{3.72}$$

has a unique solution $w \in L^2(\Omega, K_f)$, satisfying (3.31).

Now, a straight verification yields:

Theorem 5 *w is uniquely determined with respect to v by:*

$$w(x,y) = V^{kh}(y)\frac{\partial v_k}{\partial x_h}(x) + W(y)\mathrm{div}v(x). \tag{3.73}$$

By denoting

$$\beta_{ij} = \int_{Y_f} \alpha(y)(\delta_{ij} - n_i(y)n_j(y))d\sigma_y \tag{3.74}$$

$$\lambda = \int_{Y_f} \frac{\partial W_i}{\partial y_j}\frac{\partial W_i}{\partial y_j} > 0 \tag{3.75}$$

$$a_{ijkh} = \int_{Y_f}\left(\delta_{ki}\delta_{hj} + \frac{\partial V_i^{kh}}{\partial y_j}\right) + \lambda\delta_{ik}\delta_{jh} =$$

$$= \int_{Y_f}\left(\delta_{\ell k}\delta_{mh} + \frac{\partial V_\ell^{kh}}{\partial y_m}\right)\left(\delta_{\ell i}\delta_{mj} + \frac{\partial V_\ell^{ij}}{\partial y_m}\right) + \lambda\delta_{ik}\delta_{jh}, \tag{3.76}$$

by seting

$$\psi = V^{kh}\frac{\partial\varphi_k}{\partial x_h} + W\mathrm{div}\varphi \quad \text{in (3.63)} \tag{3.77}$$

and by defining

$$U = \{\theta \in L^2(\Omega\times Y_s)^N,\ \mathrm{div}_y\theta = 0 \text{ in } \Omega\times Y_s,\ \mathrm{div}\tilde{\theta} \in L^2(\Omega),\ \tilde{\theta}_n = 0 \text{ on } \partial\Omega\}, \tag{3.78}$$

$$\tilde{\theta}(x) = \frac{1}{|Y_s|}\int_{Y_s}\theta(x,y)dy, \quad \forall\theta \in L^2(\Omega\times Y_s)^N \tag{3.79}$$

$$H = \{(\theta,\varphi) \in U\times H^1_0(\Omega)^N, \theta_n = \varphi_n \text{ in } \Omega\times\Gamma,\ (1-m)\mathrm{div}\tilde{\theta} + m\mathrm{div}\varphi = 0 \text{ in } \Omega\} \tag{3.80}$$

we find, by using the corresponding density arguments

Lemma 4 *u, v and p satisfy*

$$\int_{\Omega\times Y_s} Au\theta + a_{ijkh}\int_\Omega \frac{\partial v_k}{\partial x_h}\frac{\partial\varphi_i}{\partial x_j} + \beta_{ij}\int_\Omega v_i\varphi_j = \tag{3.81}$$

$$= \int_{\Omega\times Y_s}(g - \nabla p)\theta + m\int_\Omega g\varphi, \quad \forall(\theta,\varphi) \in H$$

Remark 1 The tensor a_{ijkh} is positive definite and has the symmetry properties $a_{ijkh} = a_{khij} = a_{jikh}$. Also, we have to notice that

$$\beta_{ij} \int_{\Omega} \varphi_i \varphi_j = \int_{\Omega \times \Gamma} \alpha(\varphi - \varphi_n n)^2 \geq 0, \quad \forall \varphi \in H_0^1(\Omega)^N. \tag{3.82}$$

Next, we introduce the last two local-solutions associated to our problem. Denoting for any $i \in \{1, 2, \cdots, N\}$,

$$U_s = \{\theta \in L^2(Y_s)^N, \quad \mathrm{div}_y \theta = 0 \quad \text{in} \quad Y_s, \quad \theta_n = 0 \quad \text{on} \quad \Gamma\} \tag{3.83}$$

$$K_s^i = \{\theta \in L^2(Y_s)^N, \quad \mathrm{div}_y \theta = 0 \quad \text{in} \quad Y_s, \quad \theta_n = n_i \quad \text{on} \quad \Gamma\} \tag{3.84}$$

we define $U^i \in U_f$ and $W^i \in K_s^i$ as the unique solutions of the problems:

$$\int_{Y_s} AU^i \theta = \int_{Y_s} \theta_i, \quad \forall \theta \in U_s \tag{3.85}$$

$$\int_{Y_s} AW^i \theta = 0, \quad \forall \theta \in U_s. \tag{3.86}$$

Obviously, (3.85) is the Darcy equation in Y_s with e_i (the ith vector of the canonical basis in $\mathbb{R}^N$) as force term, while W^i is the projection of 0 on the closed convex $K_s^i \neq \emptyset$ in $V_s = \{\theta \in L^2(Y_s)^N, \ \mathrm{div}_y \theta = 0 \quad \text{in} \quad Y_s\}$.

By seting $\varphi = 0$ in (3.81) we see that, for $v \in H_0^1(\Omega)$ given, the problem

$$\int_{\Omega \times Y_s} Au\theta = \int_{\Omega \times Y_s} (g - \nabla p)\theta, \quad \forall \theta \in U_0 \tag{3.87}$$

$$U_0 = \{\theta \in U, \quad \theta_n = 0 \quad \text{in} \quad \Omega \times \Gamma\} \tag{3.88}$$

has a unique solution which satisfies the condition in H which corresponds to (3.32). Obviously, it is our u.

It is easy to verify that

Theorem 6 *u is uniquely determined with respect to p and v by*

$$u(x, y) = U^i(y) \left(g_i(x) - \frac{\partial p}{\partial x_i}(x) \right) + W^i(y) v_i(x). \tag{3.89}$$

Remark 2 The relation (3.89) is a two-scale variant of the Darcy law, which decouples the two-scale behaviour of the limit of the filtration velocity from the final homogenized system which remains to define uniquely the other two macroscopic quantities associated with the fracture: the limit of the Darcy pressure p and the limit of the Stokes velocity v. As usual, the macroscopic quantity generated by the two-scale limit of the filtration velocity is its mean value over Y_s.

By denoting

$$B_{ij} = \int_{Y_s} AW^i W^j + \beta_{ij}, \quad C_{ij} = \int_{Y_s} W_j^i, \quad D_{ij} = \int_{Y_s} U_j^i \tag{3.90}$$

and by seting

$$\theta(x, y) = W^i(y) \varphi_i(x) \quad \text{in (3.81)} \tag{3.91}$$

we find that v and p satisfy

$$B_{ij}\int_\Omega v_j\varphi_i + a_{ijkh}\int_\Omega \frac{\partial v_k}{\partial x_h}\frac{\partial \varphi_i}{\partial x_j} =$$

$$= C_{ij}\int_\Omega \left(g_j - \frac{\partial p}{\partial x_j}\right)\varphi_i + m\int_\Omega g\varphi, \quad \forall \varphi \in H_0^1(\Omega)^N \tag{3.92}$$

Taking also in account (3.28) and (3.29) we find

Theorem 7 *$(v,p) \in H_0^1(\Omega)^N \times H^1(\Omega)/\mathbb{R}$ is a weak solution of the system*

$$D_{ij}\frac{\partial}{\partial x_j}\left(g_i - \frac{\partial p}{\partial x_i}\right) + C_{ij}\frac{\partial v_i}{\partial x_j} + \left(\frac{m}{1-m}\right)\operatorname{div} v = 0 \quad in \quad \Omega \tag{3.93}$$

$$-\frac{\partial}{\partial x_j}\left(a_{ijkh}\frac{\partial v_k}{\partial x_h}\right) + B_{ij}v_j + C_{ij}\frac{\partial p}{\partial x_j} = C_{ij}g_j + mg_i \quad in \quad \Omega \tag{3.94}$$

$$D_{ij}\left(g_i - \frac{\partial p}{\partial x_i}\right)n_j = 0 \quad on \quad \partial\Omega \tag{3.95}$$

Moreover, for m sufficiently small, (v,p) is uniquely determined by this system.

Proof. As

$$C_{ij}\int_\Omega \frac{\partial v_i}{\partial x_j}p = -C_{ij}\int_\Omega \frac{\partial p}{\partial x_j}v_i$$

we define an operator on $H_0^1(\Omega)^N \times H^1(\Omega)/\mathbb{R}$ which is coercive when the influence of the last term in (3.93) is sufficiently small. ■

This final result proves that the convergences in Theorem 2 (and the followings) hold on the entire sequence, at least for m sufficiently small, and hence, the relations (3.73) and (3.88), together with the system (3.93)–(3.95) are completely describing the asymptotic behaviour of u^ε and p^ε, when $\varepsilon \to 0$.

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References

1. Adams, R. A.: Sobolev spaces, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London (1975), xviii+268.
2. Allaire, G.: Homogenization and two-scale convergence. S.I.A.M. J. Math. Anal. 23, 1482-1518 (1992)
3. Allaire, G., Murat, F.: Homogenization of the Neumann problem with non-isolated holes. Asymptotic Analysis 7, 81-95 (1993)
4. Barenblatt, G.I, Zheltov, Y.P., Kochina, I.N: On basic conceptions of the theory of homogeneous fluids seepage in fractured rocks (in Russian), Prikl.Mat. i Mekh. 24, 852-864 (1960)
5. Barenblatt, G.I., Entov, V.M., Ryzhik, V.M.: Theory of Fluid Flows Through Natural Rocks, Kluwer Acad. Pub., Dordrecht (1990)
6. Beavers, G.S., Joseph, D.D.: Boundary conditions at a naturally permeable wall. J. Fluid Mech. 30, 197–207 (1967)
7. Cioranescu, D., Saint-Jean-Paulin, J.: Homogenization in open sets with holes, J. Math. Anal. Appl. 71 (2), 590–607 (1979)

8. Ene, H.I., Poliševski, D.: Model of diffusion in partially fissured media. *Z.A.M.P.* 53, 1052–1059 (2002)
9. Jäger, W., Mikelić, A.: Modeling effective interface laws for transport phenomena between an unconfined fluid ad a porous medium using homogenization. Transp. Porous Med. 78, 489–508 (2009)
10. Jones, I.P.: Low Reynolds numer flow past a porous spherical shell. Proc Camb. Phil. Soc. 73, 231–238 (1973)
11. Lukkassen, D., Nguetseng, G. and Wall, P.: Two-scale convergence. Int.J.Pure Appl.Math. 2, 35–86 (2002)
12. Poliševski, D.: On the homogenization of fluid flows through periodic media. Rend. Sem. Mat. Univers. Politecn. Torino 45 (2), 129–139 (1987)
13. Poliševski, D.: Basic homogenization results for a biconnected ϵ-periodic structure. Appl. Anal. 82 (4), 301–309 (2003)
14. Poliševski, D.: The Regularized Diffusion in Partially Fractured Porous Media. In: L. Dragos (ed.) Current Topics in Continuum Mechanics, Volume 2, Ed. Academiei, Bucharest (2003)
15. Saffman, P.G.: On the boundary condition at the interface of a porous medium. Stud. Appl. Math. 50, 93–101 (1971)
16. Sanchez-Palencia, E.: Non-Homogeneous Media and Vibration Theory. Lecture Notes in Physics, Vol. 127, Springer-Verlag, Berlin (1980).
17. Showalter, R.E., Walkington, N.J.: Micro-structure models of diffusion in fissured media. J.Math.Anal.Appl. 155, 1-20 (1991)
18. Tartar, L.: Incompressible fluid flow in a porous medium - Convergence of the homogenization process. Appendix of [16].